ELEMENTARY LINEAR ALGEBRA – SET 6

Linear mappings and matrices

1. Define the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x + y, x)$. Show that T is linear and find the matrix $M(T)$ of the mapping T with respect to the standard basis $B = \{e_1, e_2\}$ in $\mathbb{R}^2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping given by $T(x, y) = (x + y, 3x - y)$. Find its matrix in the standard basis $B = \{e_1, e_2\}$ and in the basis $B' = \{v_1, v_2\}$ given by $v_1 = (1, 1)$, $v_2 = (1, -1)$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping given by $T(x, y, z) = (x+y, y-z, x+y+z)$. Find its matrix in the standard basis $B = \{e_1, e_2, e_3\}$.

4. Define the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x + y, x + 1)$. Show that T is not linear.

5. Define the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (xy, x)$. Show that T is not linear.

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping defined by the action onto basis vectors:

   $$T\,e_1 = e_1 + 2e_2 \quad \text{and} \quad T\,e_2 = -e_1 + e_2,$$

   where $B = \{e_1, e_2\}$ is the standard basis in $\mathbb{R}^2$. Find a formula for $T(x, y)$ for any $x, y \in \mathbb{R}$.

7. For T in Problem 6 find its matrix $M(T)$ in the standard basis $B = \{e_1, e_2\}$. Representing vectors $v = (x, y)$ in terms of one-column matrices

   $$M(v) = \begin{pmatrix} x \\ y \end{pmatrix},$$

   verify that we have $M(T)M(v) = M(T\,v)$, which gives a matrix representation of the action of T.

8. For T in Problem 6 find its matrix $M'(T)$ in the basis $B' = \{v_1, v_2\}$, where $v_1 = (1, 0)$ and $v_2 = (1, 1)$.

9. Let $T_\alpha : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping given by $T_\alpha(x, y) = (x', y')$, where

   $$\begin{aligned} x' &= x\cos\alpha - y\sin\alpha \\ y' &= x\sin\alpha + y\cos\alpha \end{aligned}$$

   where α is an arbitrary real number. This mapping represents counterclockwise rotation of the plane by α radians about the origin. Show that T_α is a linear mapping for any $\alpha \in \mathbb{R}$. Find the images of points $(1, 0), (0, 1), (-1, 0), (0, -1)$ under $T_{\pi/2}$. Find the matrix $M(T_\alpha)$ of T_α in the standard basis in $\mathbb{R}^2$.

10. Define the linear mappings that represent

(a) the reflection of the plane with respect to the y axis,

(b) the reflection of the plane with respect to the x axis,

(c) the reflection of the plane with respect to the origin.

For all these mappings find their matrices in the standard basis in $\mathbb{R}^2$.

11. Let A, B be matrices defined by

$$A = \begin{pmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 5 & 2 \\ -1 & 1 & 0 \\ -4 & 1 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 1 \end{pmatrix},$$

Which of the matrices: $A+B$, $A+C$, $2A$, AB, BA, AC, CA, A^2, B^2 are defined? Compute these matrices which are defined.

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